Decay of the classical Loschmidt echo in integrable systems

Giuliano Benenti[(1)], Giulio Casati[(1,2)], and Gregor Veble[(1,3)]
[(1)] *Center for Nonlinear and Complex Systems, Università degli Studi dell'Insubria and Istituto Nazionale per la Fisica della Materia, Unità di Como, Via Valleggio 11, 22100 Como, Italy*
[(2)] *Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Via Celoria 16, 20133 Milano, Italy*
[(3)] *Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova ul. 2, SI-2000 Maribor, Slovenia*
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We study both analytically and numerically the decay of fidelity of classical motion for integrable systems. We find that the decay can exhibit two qualitatively different behaviors, namely an *algebraic decay*, that is due to the perturbation of the shape of the tori, or a *ballistic decay*, that is associated with perturbing the frequencies of the tori. The type of decay depends on initial conditions and on the shape of the perturbation but, for small enough perturbations, not on its size. We demonstrate numerically this general behavior for the cases of the twist map, the rectangular billiard, and the kicked rotor in the almost integrable regime.

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I. INTRODUCTION

The study of the relation between classical and quantum dynamical chaos has greatly improved our understanding of the behavior of quantum systems. An issue which may be of interest in several situations is the stability of motion. In this respect, even though Liouville equation, which governs the evolution of classical distribution functions, is linear, the exponential sensitivity of classical trajectories with respect to perturbing the initial conditions leads to a strong dynamical instability, and characterizes classical chaos. Like the classical Liouville equation, the Schrödinger equation is also linear. However, the quantum evolution of states is stable and this qualitative difference is clearly apparent in the Loschmidt echo numerical experiments of Ref. [1]. The problem of the stability of quantum motion under perturbations in the Hamiltonian has recently gained a renewed interest [2–16], also in connection with quantum computation [17–19]. The quantity of central interest in these investigations is the fidelity $f_q(t)$ (also called Loschmidt echo), which measures the accuracy to which a quantum state can be recovered by inverting, at time t, the dynamics with a perturbed Hamiltonian. The main interest has been focused on classically chaotic systems for which, besides numerical experiments, some theoretical tools are available, like random matrix theory and semiclassical methods. However, in the general case, the phase space structure is mixed with chaotic components and islands of stability. If the motion starts inside an integrable island, then it very much resembles the motion in integrable systems. Contrary to chaotic systems, which are dynamically unstable but structurally stable, integrable systems are dynamically stable but very sensitive to external perturbations. Therefore the analysis of the fidelity requires particular care and one may expect it to be dependent on initial conditions and on the type of perturbation. Indeed, the decay of fidelity in integrable systems has been discussed in recent papers [11,20,21], and very different behaviors have been found. Jacquod *et al.* have shown the existence of a regime in which the quantum fidelity for classically integrable systems decays as a power law, with an anomalous exponent of purely quantum origin [21]. Prosen and Žnidarič have instead discussed a regime in which quantum fidelity exhibits a much faster Gaussian decay [11]. Both regimes have been also discussed by Eckhardt in his analysis of the decay of classical fidelity [20], in which the problem of the evolution of classical phase space densities has been addressed for linearized flows.

In the present paper, we discuss the behavior of fidelity for integrable classical systems. Besides being of interest on its own, our classical study will allow us to understand the main mechanisms for the fidelity decay and therefore will constitute a valuable reference point for the quantum analysis. Here we show the existence of a critical border depending on the *shape* of the perturbation, which separates two different types of fidelity decay: a power law decay $\propto 1/t^n$, where n is the dimension of the system, and a much faster decay of ballistic type. We stress that the type of decay depends on initial conditions and on the shape of the perturbation but, for small enough perturbations, not on its strength. We derive an analytical expression for the critical border and our theoretical results are confirmed by a numerical analysis on three different models: the twist map, the rectangular billiard and the kicked rotor.

The outline of the paper is as follows: In Section II we develop a general theory for the decay of classical fidelity in integrable systems. Section III demonstrates numerically the validity of our theory in two different typical examples of integrable systems, the twist map and the rectangular billiard, and in an almost integrable system, the kicked rotor. Our conclusions are drawn in Section IV. Finally, in the Appendix A, we discuss the long-time relaxation to equilibrium.

II. THEORY

The quantum fidelity is defined as the overlap at time t of the states $|\psi(t)\rangle$ and $|\psi_\epsilon(t)\rangle$, obtained by the evolution of the same initial state $|\psi(0)\rangle$ with the unperturbed Hamiltonian H_0 and the perturbed Hamiltonian $H_0+\epsilon V$, respectively. The fidelity is then given by

$$f_q(t) = |\langle\psi(t)|\psi_\epsilon(t)\rangle|^2 . \quad (1)$$

This expression can be equivalently rewritten in terms of the Wigner functions as

$$f_q(t) = (2\pi\hbar)^n \int d^n\mathbf{q}\; d^n\mathbf{p}\; W_\epsilon(\mathbf{q},\mathbf{p};t)\; W(\mathbf{q},\mathbf{p};t), \quad (2)$$

where n is the number of degrees of freedom. Since the Wigner functions can be considered as the quantum analogues of the classical phase space densities, we define the classical fidelity as

$$f(t) = \int d^n\mathbf{q}\; d^n\mathbf{p}\; \rho_\epsilon(\mathbf{q},\mathbf{p};t)\; \rho(\mathbf{q},\mathbf{p};t), \quad (3)$$

where ρ, ρ_ϵ are the square normalized classical phase space densities ($\int d^n\mathbf{q}\; d^n\mathbf{p}\; \rho^2 = \int d^n\mathbf{q}\; d^n\mathbf{p}\; \rho_\epsilon^2 = 1$). We note that $f(t)$ is the classical limit of $f_q(t)$. As the density evolution is unitary in both classical and quantum mechanics, instead of evolving two densities forward in time and calculating their overlap, we may first evolve the initial density ρ_0 forward in time with the unperturbed Hamiltonian H_0, and then evolve this density backward in time with the perturbed Hamiltonian $H_0 + \epsilon V$. We denote the density obtained in such a way as ρ_{2t}. The fidelity is then given by the overlap of the density ρ_{2t} with the initial density ρ_0:

$$f(t) = \int d^n\mathbf{q}\; d^n\mathbf{p}\; \rho_{2t}(\mathbf{q},\mathbf{p})\; \rho_0(\mathbf{q},\mathbf{p}). \quad (4)$$

Such an approach is more convenient for our discussion of the classical fidelity of integrable systems.

For the following discussion we assume that the perturbation of the integrable system is of the KAM type, namely that, for small enough perturbations ϵV, most of the tori of the system are only slightly deformed but not destroyed. Therefore for most of the tori the transformation from old action-angle variables $\mathbf{I}, \mathbf{\Theta}$ to new ones $\mathbf{I}', \mathbf{\Theta}'$ is possible, in such a way that the new actions are constants of motion of the perturbed system. To the first order in the perturbation strength ϵ the transformation can be written as

$$\mathbf{\Theta}' = \mathbf{\Theta} + \epsilon\mathbf{f}(\mathbf{I},\mathbf{\Theta}), \quad (5)$$
$$\mathbf{I}' = \mathbf{I} + \epsilon\mathbf{g}(\mathbf{I},\mathbf{\Theta}). \quad (6)$$

After the forward unperturbed evolution up to time t we have

$$\mathbf{\Theta}_t = \mathbf{\Theta}_0 + \mathbf{\Omega}(\mathbf{I})t. \quad (7)$$

Then we perform a backward evolution of the perturbed system from time t to time $2t$, getting

$$\mathbf{\Theta}'_{2t} = \mathbf{\Theta}'_t - \mathbf{\Omega}'(\mathbf{I}')t. \quad (8)$$

The frequency vectors $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ characterize the linear (in time) evolution of the angle variables in integrable systems. Expressed in the original action-angle variables, the overall evolution can be written as

$$\mathbf{\Theta}_{2t} = \mathbf{\Theta}_0 + (\mathbf{\Omega}(\mathbf{I}) - \mathbf{\Omega}'(\mathbf{I}'))\, t + \mathcal{O}(\epsilon), \quad (9)$$

where the error term $\mathcal{O}(\epsilon)$ is due to the change from one set of variables to the other at time t and the reverse change at time $2t$. Since the perturbation is small, we may write the change in frequency as

$$\mathbf{\Omega}'(\mathbf{I}') = \mathbf{\Omega}(\mathbf{I}) + \Delta\mathbf{\Omega}(\mathbf{I}) + \frac{\partial\mathbf{\Omega}}{\partial\mathbf{I}}(\mathbf{I}'-\mathbf{I}) + \mathcal{O}(\epsilon^2), \quad (10)$$

where $\Delta\mathbf{\Omega} \equiv \mathbf{\Omega}'(\mathbf{I}) - \mathbf{\Omega}(\mathbf{I})$ denotes the change of frequencies on the unperturbed torus $\mathbf{I}$ and $\mathbf{I}' - \mathbf{I}$ gives the change of the action variables caused by the perturbation (written in Eq. (6) to the first order in ϵ).

If we consider the angle variables to be uniformly distributed at the time t at which the motion is inverted, we may introduce the distribution

$$W_\mathbf{I}\left(\frac{\Delta\mathbf{I}}{\epsilon}\right) = \frac{1}{(2\pi)^n}\int d^n\mathbf{\Theta}\;\delta\left[\frac{\Delta\mathbf{I} - (\mathbf{I}'(\mathbf{I},\mathbf{\Theta}) - \mathbf{I}))}{\epsilon}\right], \quad (11)$$

which gives the probability density for the transition from the torus characterized by the action variables $\mathbf{I}$ in the unperturbed coordinates to the torus with action variables $\mathbf{I}'$ in the perturbed coordinates. The ϵ scaling has been chosen in order for the function $W_\mathbf{I}$ itself not to depend on ϵ in the linear approximation.

In the generic case, the motion on a torus is ergodic. It is, however, not random and therefore, in order for the equation (11) to well describe transitions between the tori, one needs to consider an ensemble of tori in the vicinity of the chosen actions $\mathbf{I}$, as only for an ensemble of tori with different frequencies we can expect the angle variables to be uniformly distributed after a sufficiently long time t. Indeed, the spread of frequencies given by $\delta\mathbf{\Omega} = \frac{\partial\mathbf{\Omega}}{\partial\mathbf{I}}\delta\mathbf{I}$ translates into the spread of angle variables $\delta\mathbf{\Theta} = \delta\mathbf{\Omega}\, t$. The time for this spread in the angle variable to become comparable to 2π is

$$t_\theta \approx \frac{2\pi}{\left(\frac{\partial\Omega}{\partial I}\right)\nu_I}, \quad (12)$$

where ν_I is the characteristic width of the initial phase space density distribution ρ_0 along the action direction (to simplify writing, we have given Eq. (12) for the one-dimensional case). Thus, our theory based on Eq. (11) is valid for times $t > t_\theta$.

Since the final angle variables, after the forward and backward evolutions, depend on the actions of the perturbed system, we compute the distribution of the angle variables from the distribution of the perturbed actions as

$$P_{\mathbf{I}}(\mathbf{\Theta}_{2t} - \mathbf{\Theta}_0; t) = W_{\mathbf{I}} \left(\frac{\mathbf{I}' - \mathbf{I}}{\epsilon} \right) \left| \frac{\partial((\mathbf{I}' - \mathbf{I})/\epsilon)}{\partial(\mathbf{\Theta}_{2t} - \mathbf{\Theta}_0)} \right|. \quad (13)$$

Using the expressions (9) and (10), we obtain

$$P_{\mathbf{I}}(\mathbf{\Theta}_{2t} - \mathbf{\Theta}_0; t) = \frac{1}{(\epsilon t)^n} \left| \frac{\partial \mathbf{\Omega}}{\partial \mathbf{I}} \right|^{-1} \times$$

$$W_{\mathbf{I}} \left(\left[\frac{\partial \mathbf{\Omega}}{\partial \mathbf{I}} \right]^{-1} \left(\frac{\mathbf{\Theta}_0 - \mathbf{\Theta}_{2t} + \mathcal{O}(\epsilon)}{\epsilon t} - \frac{\Delta \mathbf{\Omega}}{\epsilon} \right) \right). \quad (14)$$

This expression is the kernel for the combined forward and backward evolution of the phase space densities.

We assume that the width ν_I along the action direction of the initial density ρ_0 is much larger than the change of the action variable induced by the perturbation, that is

$$\nu_I \gg \epsilon \max_{I,\Theta} |g(I, \Theta)| \quad (15)$$

(to simplify writing we have given the condition (15) for the one-dimensional case). Therefore the effects of the forward and backward evolutions are felt mainly in the change of the angles variable. This means that the evolution from the initial phase space density ρ_0 to ρ_{2t} is given, up to corrections of order ϵ, by

$$\rho_{2t}(\mathbf{I}, \mathbf{\Theta}) = \int d^n \mathbf{\Theta}' \; P_{\mathbf{I}}(\mathbf{\Theta}' - \mathbf{\Theta}; t) \; \rho_0(\mathbf{I}, \mathbf{\Theta}'). \quad (16)$$

The fidelity $f(t)$ can then be computed by inserting ρ_{2t} into Eq.(4).

The kernel $P_{\mathbf{I}}(\mathbf{\Theta}' - \mathbf{\Theta})$ is stretched linearly in time, while at the same time it moves ballistically (linearly with time) with velocity $\Delta\mathbf{\Omega}$. Under the assumption that the perturbation of the shape of the tori is not divergent (as it is the case for most of the tori in a KAM regime), the distribution $W_{\mathbf{I}}(\mathbf{I}/\epsilon)$ has a bounded support which is determined by the change of the shape of the tori due to the perturbation. We can see from Eq. (14) that at long times the argument of the function $W_{\mathbf{I}}$ is given by $-[\partial\mathbf{\Omega}/\partial\mathbf{I}]^{-1}(\Delta\mathbf{\Omega}/\epsilon)$. Therefore the long time behavior of $P_{\mathbf{I}}$ depends on whether the value of $-[\partial\mathbf{\Omega}/\partial\mathbf{I}]^{-1}(\Delta\mathbf{\Omega}/\epsilon)$ falls within the support of $W_{\mathbf{I}}$ or not. In the first case, $\tilde{W}_{\mathbf{I}} \equiv W_{\mathbf{I}}(-[\partial\mathbf{\Omega}/\partial\mathbf{I}]^{-1}(\Delta\mathbf{\Omega}/\epsilon))$ is different from zero and therefore the kernel $P_{\mathbf{I}}$ drops $\propto 1/t^n$. In the latter case, $\tilde{W}_{\mathbf{I}} = 0$, and therefore $P_{\mathbf{I}}$ drops ballistically. The transition between these two regimes is determined by the equality

$$\frac{\Delta \mathbf{I}_s}{\epsilon} = - \left[\frac{\partial \mathbf{\Omega}}{\partial \mathbf{I}} \right]^{-1} \frac{\Delta \mathbf{\Omega}}{\epsilon}, \quad (17)$$

where $\Delta\mathbf{I}_s/\epsilon$ are the coordinates of the border of the support of $W_{\mathbf{I}}$.

We can therefore draw the following conclusions: If the perturbation of a classical integrable system is such that the primary effect is the change of the shape of the tori, then the expected decay of fidelity is $\propto 1/t^n$. On the contrary, if the change of the frequencies of the tori is the dominant effect, then we expect a ballistic decay of fidelity, that is the center of mass motion of the phase space densities after the forward and backward evolutions is responsible for a drastic drop of fidelity. Such a decay takes place as soon as the centers of mass of the densities ρ_{2t} and ρ_0 are separated in the angle variables $\mathbf{\Theta}$ by more than their characteristic width $\nu_{\mathbf{\Theta}}$. As it can be seen from Eq. (4), the exact form of the fidelity drop in the ballistic regime depends on the tails of the initial distribution ρ_0: for instance, a Gaussian tail gives a Gaussian decay of fidelity, whereas a sharp border induces a sharp drop to zero of fidelity. Finally, it is important to stress that the type of decay, power law or ballistic, depends on initial conditions and on the shape of the perturbation. However, it does not depend on the strength of the perturbation, provided that it is sufficiently small. Indeed, Eq. (17) shows that $\Delta\mathbf{I}_s/\epsilon$ is ϵ-independent (to the first order in ϵ), since $\Delta\mathbf{\Omega} \propto \epsilon$.

III. NUMERICAL DEMONSTRATION

As a first example we consider the perturbed twist map, defined by

$$\begin{aligned} I_{t+1} &= I_t + \epsilon \cos(\alpha) \sin(\Theta_t), \\ \Theta_{t+1} &= \Theta_t + I_{t+1} + \epsilon \sin(\alpha) \sin(I_{t+1}), \end{aligned} \quad (18)$$

where the angle α determines the mixture between purely perturbing the shape of the tori ($\alpha = 0$) or purely changing their frequencies ($\alpha = \pi/2$). This parametrization allows us to change the type of the perturbation without changing its overall magnitude. The change of frequency associated with the perturbation is given by

$$\Delta\Omega = \epsilon \sin(\alpha) \sin(I). \quad (19)$$

The conserved action variable of the ϵ-perturbed system is, to the first order (in ϵ) approximation, given by

$$I' = I + \epsilon \cos(\alpha) \frac{1}{2\sin(I/2)} \cos\left(\Theta - \frac{I}{2}\right). \quad (20)$$

Indeed, inserting this expression into the mapping (18), one can easily verify that $I_{t+1} = I_t$. The transition probability function W_I for this system can thus be obtained by means of Eq. (11):

$$W_I\left(\Delta I/\epsilon\right) =$$

$$\frac{1}{2\pi} \int d\Theta \; \delta\left[\frac{\Delta I}{\epsilon} - \frac{\cos(\alpha)}{2\sin(I/2)} \cos\left(\Theta - \frac{I}{2}\right)\right], \quad (21)$$

which gives

$$W_I\left(\frac{\Delta I}{\epsilon}\right)=\frac{1}{\pi\sqrt{\left(\frac{\cos(\alpha)}{2\sin(I/2)}\right)^2-(\Delta I/\epsilon)^2}}. \tag{22}$$

The range of the support of the distribution W_I is between $\pm\cos(\alpha)/(2\sin(I/2))$. The critical value α_c for which $\Delta I_s/\epsilon = -(\partial\Omega/\partial I)^{-1}\Delta\Omega/\epsilon$ is therefore determined by

$$\tan(\alpha_c)=\frac{1}{2\sin(I)\sin(I/2)}. \tag{23}$$

In Fig. 1 we show the numerically computed behavior of fidelity for this system as a function of time for various values of the parameter α. We take as initial phase space density a rectangle centered around the point $(\Theta=\pi, I=1)$ with sides of length $\nu_\Theta = 2\times10^{-3}$, $\nu_I = 2\times10^{-2}$. The perturbation strength is $\epsilon = 10^{-6}$. To compute fidelity, we follow the evolution of $N = 10^4$ trajectories, which at $t = 0$ are uniformly distributed inside the above rectangle. The fidelity is then given by the percentage of trajectories that return back to this region after the forward and backward evolutions. In all cases we observe an initial plateau during which the fidelity does not decay appreciably. This plateau persists until the time t_p at which the width of the kernel (14) or the shift of its center become comparable to the width ν_Θ of the phase space density along the angle variable. In either case this time is

$$t_p \propto \frac{\nu_\Theta}{\epsilon}. \tag{24}$$

According to Eq. (17), we expect the behavior to change from algebraic decay to ballistic one at the value of the parameter $\alpha = \alpha_c$. For the chosen initial conditions, Eq. (23) gives $\alpha_c \approx 0.892$. Indeed, the change from an algebraic fidelity decay $f(t) \propto 1/t$ when $\alpha < \alpha_c$ to a sharp drop of fidelity when $\alpha > \alpha_c$ is clearly seen in Fig. 1.

An interesting feature is that, approaching the critical value α_c, we observe that the fidelity decay, power law or ballistic, sets in after longer and longer times. This fact has a clear explanation: The value $\tilde{W}_{\mathbf{I}} = W_I(-[\partial\mathbf{\Omega}/\partial\mathbf{I}]^{-1}(\Delta\mathbf{\Omega}/\epsilon))$, which determines the long time behavior of the evolution kernel (14), diverges close to the critical value $\alpha = \alpha_c$ (see Eq. (22)). When the fidelity decay is power law, we have $P_{\mathbf{I}} = c/t^n$, with the constant $c \propto W_{\mathbf{I}}$. Since c becomes larger and larger close to the critical point, the fidelity decay must be postponed to longer and longer times. On the other hand, when the long time fidelity decay is ballistic, we can see from Eq. (14) that the argument of the function $W_{\mathbf{I}}$ goes outside the support of $W_{\mathbf{I}}$ after a time that becomes longer close to the critical point. Only after this time the fidelity drops off. Of course the decay cannot be postponed indefinitely since the exact condition (23) can be satisfied only for a single torus, while we always deal with a family of tori upon which the initial phase space density ρ_0 rests.

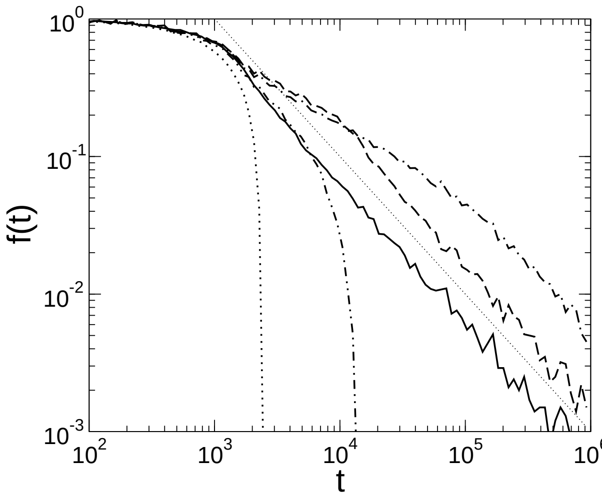

FIG. 1. Fidelity decay for the twist map at various values of the parameter $\alpha = 0$ (full line), 0.8 (dashed), 0.892 (dot-dashed), 1.0 (dot-dot-dashed) and $\pi/2$ (dotted). The $\propto 1/t$ decay is shown as a thin dotted line

We also checked numerically that, provided that the perturbation is much smaller than the characteristic widths ν_I of the initial density (that is, the requirement (15) is fulfilled), the type of behavior does not alter with changing the actual size of the perturbation ϵ, as it is expected from our theory. We simply rescale the time $t_p \propto 1/\epsilon$ after which the fidelity decay starts, in agreement with Eq. (24).

To illustrate the fidelity decay in integrable systems with more than one degree of freedom, we consider the following system:

$$H(I_1,I_2,\Theta_1,\Theta_2) = H_0(I_1,I_2)+\epsilon V(I_1,I_2,\Theta_1,\Theta_2), \tag{25}$$

where the unperturbed Hamiltonian

$$H_0=\frac{\alpha_1}{2}I_1^2+\frac{\alpha_2}{2}I_2^2 \tag{26}$$

describes the motion of a particle bouncing elastically inside a rectangular billiard and the perturbation is given by

$$V=\cos(\beta)\cos(\Theta_1)\cos(\Theta_2)+\sin(\beta)I_1I_2. \tag{27}$$

Again, depending on the value of the parameter β, the perturbation mainly affects either the shape of the tori or their frequencies. We use the first order perturbation theory of Hamiltonian systems (see, e.g., Ref. [22]) to determine the effects of the perturbation. What we need to find is a set of action-angle coordinates such that, to the first order in the perturbation strength ϵ, the Hamiltonian (25) in these new coordinates can be written as a function of only the new actions, namely

$$H(I_1,I_2,\Theta_1,\Theta_2)=H'(I_1',I_2')+\mathcal{O}(\epsilon^2). \tag{28}$$

Introducing the generating function

$$G(I_1', I_2', \Theta_1, \Theta_2) = I_1'\Theta_1 + I_2'\Theta_2$$
$$-\frac{\epsilon\cos\beta}{2}\left[\frac{\sin(\Theta_1+\Theta_2)}{\alpha_1 I_1' + \alpha_2 I_2'} + \frac{\sin(\Theta_1-\Theta_2)}{\alpha_1 I_1' - \alpha_2 I_2'}\right], \tag{29}$$

we get

$$I_1 = \frac{\partial G}{\partial \theta_1} = I_1' - \frac{\epsilon\cos(\beta)}{2}\left[\frac{1}{\alpha_1 I_1 + \alpha_2 I_2}\cos(\Theta_1+\Theta_2)\right.$$
$$\left.+\frac{1}{\alpha_1 I_1 - \alpha_2 I_2}\cos(\Theta_1-\Theta_2)\right], \tag{30}$$

$$I_2 = \frac{\partial G}{\partial \theta_2} = I_2' - \frac{\epsilon\cos(\beta)}{2}\left[\frac{1}{\alpha_1 I_1 + \alpha_2 I_2}\cos(\Theta_1+\Theta_2)\right.$$
$$\left.-\frac{1}{\alpha_1 I_1 - \alpha_2 I_2}\cos(\Theta_1-\Theta_2)\right]. \tag{31}$$

Substituting the above expressions into the Hamiltonian (25), we get

$$H'(I_1', I_2') = \frac{\alpha_1^2}{2}{I_1'}^2 + \frac{\alpha_2^2}{2}{I_2'}^2 + \epsilon\sin(\beta)I_1'I_2'. \tag{32}$$

The new frequencies are then given by

$$\Omega_1' = \frac{\partial H'}{\partial I_1'} = \alpha_1 I_1' + \epsilon\sin(\beta)I_2', \tag{33}$$

$$\Omega_2' = \frac{\partial H'}{\partial I_2'} = \alpha_2 I_2' + \epsilon\sin(\beta)I_1'. \tag{34}$$

Thus the frequencies changes read as follows:

$$\Delta\Omega_1 = \epsilon\sin(\beta)I_2, \tag{35}$$

$$\Delta\Omega_2 = \epsilon\sin(\beta)I_1. \tag{36}$$

As in the previous example, the above expressions allow us to find the transition probability function

$$W_{\mathbf{I}}(\Delta I_1/\epsilon, \Delta I_2/\epsilon) =$$
$$\frac{2}{\pi^2}\frac{1}{\sqrt{\left(\frac{\cos(\beta)}{\alpha_1 I_1+\alpha_2 I_2}\right)^2 - \left(\frac{\Delta I_1 + \Delta I_2}{\epsilon}\right)^2}}\times$$
$$\frac{1}{\sqrt{\left(\frac{\cos(\beta)}{\alpha_1 I_1-\alpha_2 I_2}\right)^2 - \left(\frac{\Delta I_1 - \Delta I_2}{\epsilon}\right)^2}}. \tag{37}$$

It can be seen that the support for the distribution $W_{\mathbf{I}}$ is the rectangle

$$|U| < \frac{\cos(\beta)}{\alpha_1 I_1 + \alpha_2 I_2}, \quad |V| < \frac{\cos(\beta)}{\alpha_1 I_1 - \alpha_2 I_2}, \tag{38}$$

where $U = (\Delta I_1 + \Delta I_2)/\epsilon$ and $V = (\Delta I_1 - \Delta I_2)/\epsilon$.

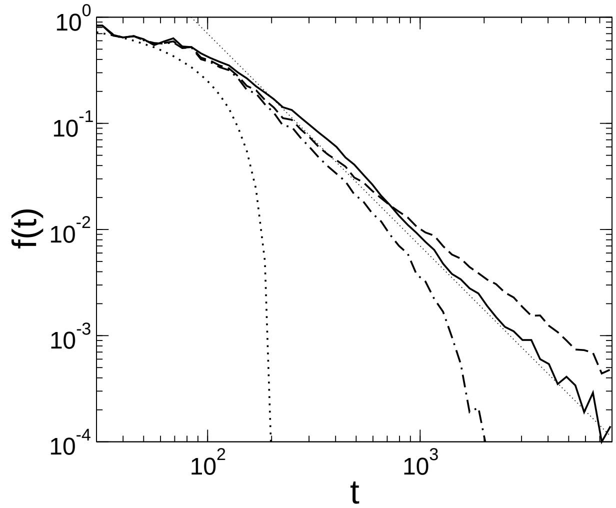

FIG. 2. Fidelity decay for the rectangular billiard for various values of the parameter $\beta = 0$ (full line), 0.232 (dashed), 0.3 (dot-dashed) and $\pi/2$ (dotted). The $\propto 1/t^2$ decay is shown as a thin dotted line.

In Fig. 2 we show the decay of fidelity for this system for various values of β. The parameters of the system have been chosen as follows: $\alpha_1 = (\sqrt{5}+1)/2$, $\alpha_2 = 1$. In all cases the initial phase space density is a hyper-rectangle centered around $I_1 = 1$, $I_2 = 1$, $\Theta_1 = 1$ and $\Theta_2 = 1$ with all sides of length $\nu_{I_1} = \nu_{I_1} = \nu_{\Theta_1} = \nu_{\Theta_2} = 0.02$. The perturbation parameter is $\epsilon = 3\times 10^{-4}$ and the number of trajectories $N = 10^5$. For the above initial conditions and parameters α_1, α_2, the critical value β_c which separates the power law and the ballistic fidelity decay is determined by the equality (17) in the direction of the U variable. Indeed, when β increases, $-[\partial\mathbf{\Omega}/\partial\mathbf{I}]^{-1}(\Delta\mathbf{\Omega}/\epsilon)$ goes outside the support of $W_{\mathbf{I}}$ at first along this direction. This gives

$$-\left(\frac{\Delta\mathbf{I}_s}{\epsilon}\right)_U = \frac{\cos(\beta_c)}{\alpha_1 I_1 + \alpha_2 I_2} = \left[\left(\frac{\partial\mathbf{\Omega}}{\partial\mathbf{I}}\right)^{-1}\Delta\mathbf{\Omega}\right]_U, \tag{39}$$

where the right-hand side is the U-component of the vector

$$\left(\frac{\partial\mathbf{\Omega}}{\partial\mathbf{I}}\right)^{-1}\Delta\mathbf{\Omega} = \begin{pmatrix}\alpha_2^{-1} & 0\\ 0 & \alpha_1^{-1}\end{pmatrix}\begin{pmatrix}\sin(\beta_c)I_2\\ \sin(\beta_c)I_1\end{pmatrix}. \tag{40}$$

Therefore we get

$$\tan(\beta_c) = \frac{\alpha_1\alpha_2}{(\alpha_1 I_1 + \alpha_2 I_2)^2}. \tag{41}$$

Substituting the chosen values of $I_1, I_2, \alpha_1,$ and α_2, we find that the critical value is equal to $\beta_c \approx 0.232$. This theoretical expectation is confirmed by the numerical data of Fig. 2, which show a crossover from a power law fidelity decay (for $\beta < \beta_c$) to a ballistic decay (for $\beta > \beta_c$).

The results are very similar to the case of the twist map, including the fact that, close to the critical value $\beta = \beta_c$, the decay is postponed to longer times. Is should be stressed that the algebraic fidelity decay, differently from the twist map case, is now inversely proportional to the square of the time, in agreement with our theoretical expectation for a two-dimensional system (see Eq. (14)).

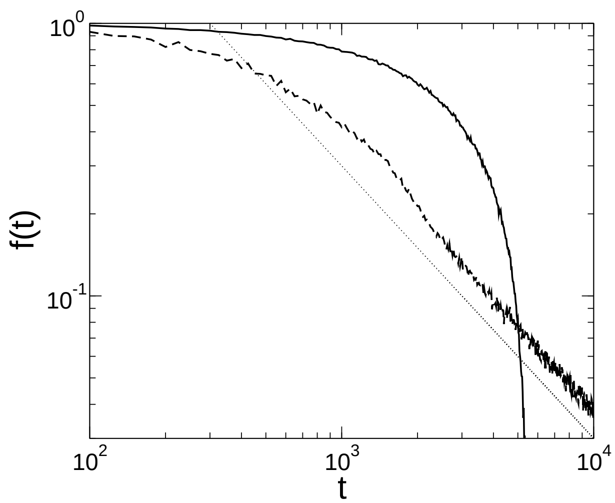

FIG. 3. Fidelity decay for the kicked rotor map with $K = 0.3$, $\epsilon = K' - K = 10^{-5}$, and $N = 10^4$ trajectories, using two different initial phase space densities, one centered at the point $(\theta = \pi, I = 0.2)$ (full line) and the other centered at $(\theta = \pi, I = 1.2)$ (dashed line). In the first case the fidelity decays ballistically, in the latter case inversely proportional to time. Both initial densities are 0.02 wide in the Θ and I directions. The $\propto 1/t$ curve is shown as a thin dotted line.

As a last numerical example, we consider the kicked rotor map that is given by

$$I_{t+1} = I_t + K \sin(\Theta_t), \tag{42}$$

$$\Theta_{t+1} = \Theta_t + I_{t+1}. \tag{43}$$

As it is known, for $K \ll 1$ the system is almost integrable, namely its phase space is dominated by invariant tori. There is a stable fixed point at $(\Theta = \pi, I = 0)$ and a separatrix which divides the phase space into two regions: a section of librational motion around the stable fixed point inside the separatrix, and a section of rotational motion outside the separatrix. We perturb the system by varying $K \to K' = K + \epsilon$. The important point is that the type of the perturbation chosen strongly affects the frequencies of the tori in the librational section, while it mainly perturbs the shape of the rotational tori. Therefore the same system and perturbation should lead to two completely different types of fidelity decays, power law or ballistic, depending on the choice of the initial conditions. Fig. 3 confirms this expectation: if the initial density ρ_0 is inside the separatrix the fidelity decay is ballistic, otherwise it is power law.

IV. CONCLUSIONS AND OUTLOOK

In this work we have studied the decay of the fidelity of classical motion for integrable systems. Our main result is the following: for small enough perturbations, the type of the decay of fidelity for integrable systems depends not on the strength of the perturbation but on its shape and on initial conditions. More precisely, the fidelity exhibits two completely different behaviors, namely an algebraic decay if the perturbation mainly affects the shape of the tori, and a faster, ballistic decay, if the main effect of the perturbation is to change the frequencies of the tori. We have also given clear numerical demonstrations of the transition between the two types of behaviors, induced by changing the shape of the perturbation or the initial conditions.

This result poses interesting questions with respect to the quantum mechanical picture. Due to the correspondence principle, there should exist regimes where both types of decay may be observed. It is however expected that, for small perturbations, quantum mechanics would favor the ballistic type decay, as demonstrated in [11]. Indeed the algebraic decay is due to the transitions between tori which, for small perturbations, are suppressed in quantum mechanics, due to tori quantization and subsequent gaps between them. The classical-quantum correspondence will be the topic of further studies.

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APPENDIX A: ASYMPTOTIC BEHAVIOR

The results of the previous Sections do not tackle the asymptotic decay of fidelity for integrable systems [23]. Indeed, we neglected the contributions to the evolution kernel (14) that stem from the fact that the angle variables are cyclic. This means that, after a time which is $\propto 2\pi/\epsilon$, we need to take into account the contributions to (14) not only at $\mathbf{\Theta}_{2t} - \mathbf{\Theta}_0$ but also at all $\mathbf{\Theta}_{2t} - \mathbf{\Theta}_0 + 2\pi\mathbf{k}$, where $\mathbf{k}$ is a vector of integer numbers.

We limit ourselves to the case of a single torus. Of course the fidelity $f(t)$ is strictly zero for a single torus, and therefore it should be understood that we take the limits

$$\epsilon \to 0, \;\; \nu_{\mathbf{I}} \to 0, \;\; \text{with} \;\; \frac{\epsilon}{\nu_{\mathbf{I}}} = \text{constant} \ll 1. \tag{A1}$$

Let us consider the initial density $\rho_0(\mathbf{\Theta})$ to be defined on the whole $\mathbf{\Theta}$ space (without 2π periodicity), while the kernel $K_{\mathbf{I}}(\mathbf{\Theta}; t)$ is defined as the periodic function obtained from the original kernel:

$$K_{\mathbf{I}}(\mathbf{\Theta};t)=\sum_{\mathbf{k}}P_{\mathbf{I}}(\mathbf{\Theta}-2\pi\mathbf{k};t) \quad (A2)$$

Assuming that the initial density is square normalized, the fidelity can be written as

$$f(t)=\int d^n\mathbf{\Theta}\rho_0^*(\mathbf{\Theta})\rho_{2t}(\mathbf{\Theta})=\int d^n\mathbf{\Phi}\tilde{\rho}_0^*(\mathbf{\Phi})\tilde{\rho}_{2t}(\mathbf{\Phi}), \quad (A3)$$

where ˜ denotes the Fourier transform:

$$\tilde{\rho}(\mathbf{\Phi})=\frac{1}{\sqrt{2\pi}}\int d^n\mathbf{\Theta}\rho(\mathbf{\Theta})\exp(-i\mathbf{\Phi\Theta}). \quad (A4)$$

Since

$$\rho_{2t}(\mathbf{\Theta})=\int d^n\mathbf{\Theta}'\rho_0(\mathbf{\Theta}')K_{\mathbf{I}}(\mathbf{\Theta}-\mathbf{\Theta}';t), \quad (A5)$$

in the Fourier picture this becomes

$$\tilde{\rho}_{2t}(\mathbf{\Phi})=\tilde{\rho}_0(\mathbf{\Phi})\tilde{K}_{\mathbf{I}}(\mathbf{\Phi};t). \quad (A6)$$

We may write the original kernel (14) in the simplified form

$$P_{\mathbf{I}}(\mathbf{\Theta};t)=1/t^n p_{\mathbf{I}}(\mathbf{\Theta}/t+\mathbf{\Gamma}), \quad (A7)$$

where $p_{\mathbf{I}}(\mathbf{y}) = \epsilon^{-n}W_{\mathbf{I}}\left(-\left[\frac{\partial\mathbf{\Omega}}{\partial\mathbf{I}}\right]^{-1}\frac{\mathbf{y}}{\epsilon}\right)\left|\frac{\partial\mathbf{\Omega}}{\partial\mathbf{I}}\right|^{-1}$ and $\mathbf{\Gamma} = \Delta\mathbf{\Omega}$. The Fourier transform of the kernel (A2) is therefore given by

$$\tilde{K}_{\mathbf{I}}(\mathbf{\Phi};t)=\sum_{\mathbf{k}}\tilde{p}_{\mathbf{I}}(t\mathbf{\Phi})\exp(i\mathbf{\Phi}(\mathbf{\Gamma}t-2\pi\mathbf{k})). \quad (A8)$$

Then the formula

$$\sum_{\mathbf{k}}\exp(-i2\pi\mathbf{\Phi k})=\sum_{\mathbf{j}}\delta(\mathbf{\Phi}-\mathbf{j}) \quad (A9)$$

leads to

$$\tilde{K}_{\mathbf{I}}(\mathbf{\Phi};t)=\sum_{\mathbf{j}}\tilde{p}_{\mathbf{I}}(t\mathbf{j})\ \exp(it\mathbf{\Gamma j})\ \delta(\mathbf{\Phi}-\mathbf{j}). \quad (A10)$$

This result, coupled with equations (A3) and (A6), finally leads to

$$f(t)=\sum_{\mathbf{j}}|\tilde{\rho}_0(\mathbf{j})|^2\ \tilde{p}_{\mathbf{I}}(t\mathbf{j})\ \exp(it\mathbf{\Gamma j}). \quad (A11)$$

As we can see, the behavior of fidelity in the limit $t\to\infty$ is given by the tails of the Fourier transform of the kernel $p_{\mathbf{I}}$. The origin of the kernel is the projection of the perturbed tori onto unperturbed ones, and we expect singularities in such a projection. These singularities induce a power law decay in the tails of the Fourier transform of the kernel and thus are responsible for the asymptotic power law decay of fidelity.

In the single degree of freedom situation, the typical singularity of projection to be encountered leads to $p_I(y)\propto|y-y_0|^{-1/2}$, as it can also be seen in the twist map example (22). This type of singularity leads to the Fourier transform

$$\tilde{p}_I(\Phi)\propto\Phi^{-1/2}\exp(-i\Phi y_0). \quad (A12)$$

Such an expression leads to following asymptotic fidelity decay:

$$f(t)-f(\infty)=$$

$$t^{-1/2}\sum_{j\neq0}|\tilde{\rho}_0(j)|^2\ j^{-1/2}\exp(ij(\Gamma t-y_0))=t^{-1/2}z(\beta), \quad (A13)$$

where $\beta = -\Gamma t + y_0$ and z is some periodic function with period 2π. We note that Eq. (A13) gives an overall $\propto t^{-1/2}$ fidelity decay together with a superimposed oscillatory behavior. This is the typical asymptotic relaxation of fidelity for a single torus in integrable systems with a single degree of freedom. If one considers a finite interval of actions ν_I, the decay (A13) must be averaged over ν_I, and therefore, due to the oscillatory nature of Eq. (A13), it can be faster that $t^{-1/2}$. The extension to the many-dimensional case requires a complex analysis of the singularities encountered in the projection of the perturbed tori onto the unperturbed ones and is beyond the scope of the present paper.

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